NUMERICAL SOLUTION OF THE QUASISTATIONARY AXISYMMETRIC STEFAN PROBLEM

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The problem of melting of a cylindrical rod moving at constant velocity through a heat source is solved by mesh methods.

Many heat and mass transfer problems are associated with the need to study thermal processes with a phase transition taken into account. In this respect, we note problems of the crystallization and melting of solid substances, for instance. In their nature, the mathematical models of thermal processes with a transition from one phase state to another are nonlinear and multidimensional. It is impossible to investigate them successfully without using calculation methods [1]. Many papers are devoted to the numerical solution of problems with a phase transition. In particular, the method with smoothing of the coefficients is used most extensively for multidimensional nonstationary Stefan problems [2, 3]. Another approach (methods with an explicit extraction of the phase transition boundary) is applied mainly in one-dimensional problems [4].

The quasistationary Stefan problem in which the phase interface is fixed or moves at a constant rate can be extracted out of the total spectrum of problems with a change in the phase state. Certain mathematical questions about the existence and uniqueness of solutions of this problem are investigated in the simplest formulation in [5].

Results of a numerical solution of the quasistationary axisymmetric Stefan problem, which can be considered a model for describing thermal processes in the continuous teeming of steel and the zone melting of crystals [6], are presented in this paper. A method, developed by the authors of this paper and based on application of potential theory, is used. The same model problem was examined in [7], where a build-up method with smoothing of the coefficients was used. Results are presented of computations with different dimensionless parameters characterizing the problem.

FORMULATION OF THE PROBLEM

A cylindrical rod of radius R moves past a heat source at the constant velocity v_0. The thermal process with a phase transition from one state of the substance to another is described in the variables (r, z) by the following equation for the temperature:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rk(u)\frac{\partial u}{\partial r}\right)+\frac{\partial}{\partial z}\left(k(u)\frac{\partial u}{\partial z}\right)=c(u)v_0\frac{\partial u}{\partial z}. \quad (1)$$

The Stefan condition

$$\left(k(u)\frac{\partial u}{\partial n}\right)_1-\left(k(u)\frac{\partial u}{\partial n}\right)_2=-\lambda v_0\cos(n,\ z) \quad (2)$$

is satisfied on the phase interface $S(u = u^*)$. The heat-conduction coefficient and specific heat in (1) and (2) are assumed discontinuous in the general case:

$$k(u)=\begin{cases}k_1(u), & u<u^*,\\ k_2(u), & u\geqslant u^*,\end{cases}\quad c(u)=\begin{cases}c_1(u), & u<u^*,\\ c_2(u), & u\geqslant u^*.\end{cases}$$

For definiteness, n is the external normal to S with respect to the domain D_1 ($u < u^*$, solid phase), while $u > u^*$ in D_2 (liquid phase), and cos(n, z) is the cosine of the angle between the normal n and the axis OZ.

The influence of the heat source is felt on a certain section of the rod $0 \le z \le L$. Let us extract a rectangle of length L and width R and let us examine the problem in this

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domain. We consider the temperature constant and equal to the environment temperature u_0 on the sides of the rectangle $D=\{(r, z)|0<r<R, 0<z<L\}$:

$$u(r, 0)=u(r, L)=u_0. \tag{3}$$

The heat flux

$$k(u)\frac{\partial u}{\partial r}(R, z)=Q(z) \tag{4}$$

is given on the upper boundary for r = R, while the condition

$$rk(u)\frac{\partial u}{\partial r}(r, z)\to 0, \ r\to 0 \tag{5}$$

is satisfied for r = 0. Using the same notation for the dimensionless variables as for the dimensional quantities, we obtain the following problem from (1)-(5):

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rk(u)\frac{\partial u}{\partial r}\right)+\frac{\partial}{\partial z}k(u)\frac{\partial u}{\partial z}\bigg)=\mathrm{Pe}\frac{\partial u}{\partial z}, \tag{6}$$

$$\left(k(u)\frac{\partial u}{\partial n}\right)_1-\left(k(u)\frac{\partial u}{\partial n}\right)_2=-\mathrm{Pe}\,\mathrm{St}\cos(n, z), \ u=1, \tag{7}$$

$$u(r, 0)=u(r, L)=u_0, \tag{8}$$

$$k(u)\frac{\partial u}{\partial r}(1, z)=\mathrm{Kr}\,Q(z), \tag{9}$$

$$rk(u)\frac{\partial u}{\partial r}(r, z)\to 0, \ r\to 0. \tag{10}$$

The problem (6)-(10) is characterized by the dimensionless parameters Pe, St, and Kr.

METHOD OF SOLUTION

We use the method of additive extraction of the jump in the normal derivatives (7) on the unknown phase interface S governing the specifics of the Stefan problem. We first formulate the problem for a new auxiliary function

$$v(u)=\int_0^u k(\xi)\,d\xi.$$

Equation (6) and conditions (7)-(10) yield the following problem for v:

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial v}{\partial r}+\frac{\partial^2 v}{\partial z^2}=\mathrm{Pe}\,\varkappa(v)\frac{\partial v}{\partial z}, \tag{11}$$

$$\left(\frac{\partial v}{\partial n}\right)_1-\left(\frac{\partial v}{\partial n}\right)_2=-\mathrm{Pe}\,\mathrm{St}\cos(n, z), \ v=v^*, \tag{12}$$

$$v(r, 0)=v(r, L)=\eta_0, \tag{13}$$

$$\frac{\partial v}{\partial r}(1, z)=\mathrm{Kr}\,Q(z), \tag{14}$$

$$r\frac{\partial v}{\partial r}(r, z)\to 0, \ r\to 0. \tag{15}$$

Here

$$\varkappa(v)=c(u(v))/k(u(v)), \ v^*=\int_0^1 k(\xi)\,d\xi, \ \eta_0=\int_0^{u_0} k(\xi)\,d\xi.$$

We seek the solution of the problem (11)-(15) for v in the form of the sum of two functions

$$v(r, z)=V(r, z)+\omega(r, z), \tag{16}$$

where $\omega(r, z)$ is a continuous function together with its first derivatives in the whole domain D while $V(r, z)$ is the potential of a simple layer,

$$V(r, z) = \int_S \nu(r', z') G(r, z; r', z') dl. \tag{17}$$

Here $G(r, z; r', z')$ is the fundamental solution of the Laplace equation in the axisymmetric case, which is described as follows:

$$G(r, z; r', z') = \frac{1}{2\pi} t \sqrt{\frac{r'}{r}} K(t). \tag{18}$$

In (18)

$$t^2 = \frac{4rr'}{(r + r')^2 + (z - z')^2},$$

$$K(t) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - t^2 \sin^2 \varphi}}$$

is the complete elliptical integral of the first kind. We note that the normal derivative of the simple-layer potential (17), (18) undergoes a discontinuity on S whose magnitude is determined by the value of the density at the point of discontinuity [8]:

$$\left(\frac{\partial V}{\partial n}\right)_1 - \left(\frac{\partial V}{\partial n}\right)_2 = \nu(r, z).$$

Setting

$$\nu(r, z) = -\mathrm{Pe}\,\mathrm{St} \cos(n, z), \tag{19}$$

we satisfy the conjugate condition (12) for the function $v(r, z)$ by using (16) and (17). The function $\omega(r, z)$ has no singularities in D for such an extraction of the jump in the normal derivative (12), and is a solution of the equation

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \omega}{\partial r} + \frac{\partial^2 \omega}{\partial z^2} = \mathrm{Pe}\,\varkappa(\omega + V) \frac{\partial (\omega + V)}{\partial z}, \tag{20}$$

where the potential $V(r, z)$ is determined by (17)-(19). Boundary conditions for $\omega(r, z)$ result from (13)-(16):

$$\omega(r, 0) = \eta_0 - V(r, 0), \quad \omega(r, L) = \eta_0 - V(r, L), \tag{21}$$

$$\frac{\partial \omega}{\partial r}(1, z) = \mathrm{Kr}\,Q(z) - \frac{\partial V}{\partial r}(1, z), \tag{22}$$

$$r \frac{\partial \omega}{\partial r}(r, z) \to 0, \quad r \to 0. \tag{23}$$

NUMERICAL REALIZATION OF THE METHOD

We use an iteration process of successive refinement of the unknown phase interface, performed in several stages, to solve the problem (16)-(23) numerically.

1) Let there be the k-th approximation for the function $v(r, z)$ in the domain D, where $\overset{0}{v}(r, z)$ is given sufficiently arbitrarily. We denote the appropriate approximation for the phase interface by $\overset{k}{S}$, where $\overset{k}{S} = \{(r, z) \mid \overset{k}{v}(r, z) = v^*\}$. Then in conformity with (17) we have

$$\overset{k}{V}(r, z) = \int_{\overset{k}{S}} \overset{k}{\nu}(r', z') G(r, z; r', z') dl,$$

and the density $\overset{k}{\nu}(r, z)$ is determined according to (19).

2) The next (k + 1)-th approximation for the auxiliary function $\omega(r, z)$ is determined from the solution of the linear problem

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \overset{k+1}{\omega}}{\partial r} + \frac{\partial^2 \overset{k+1}{\omega}}{\partial z^2} = \mathrm{Pe}\,\varkappa(\overset{k}{v}) \frac{\partial (\overset{k+1}{\omega} + \overset{k}{V})}{\partial z}, \tag{24}$$

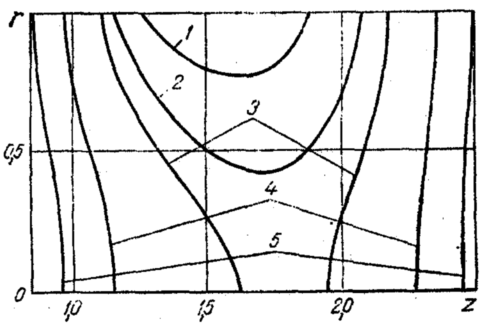

Fig. 1.

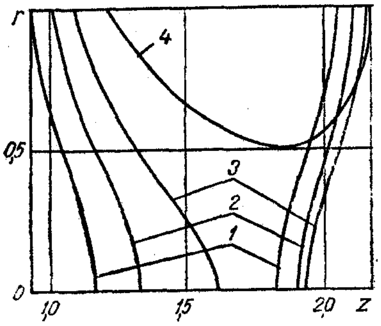

Fig. 2.

Fig. 1. Melt boundary for different values of the Kirpichev number (St = 0.25; Pe = 0.5): 1) Kr = 0.75; 2) 0.9; 3) 1.0; 4) 1.25; 5) 1.5.

Fig. 2. Influence of the Peclet number on the melt boundary (St = 0.25, Kr = 1.0): 1) Pe = 0; 2) 0.25; 3) 0.5; 4) 1.0.

$$\overset{k+1}{\omega}(r,\ 0) = \eta_0 - \overset{k}{V}(r,\ 0),\quad \overset{k+1}{\omega}(r,\ L) = \eta_0 - \overset{k}{V}(r,\ L), \tag{25}$$

$$\frac{\partial \overset{k+1}{\omega}}{\partial r}(1,\ z) = \mathrm{Kr}\, Q(z) - \frac{\partial \overset{k}{V}}{\partial z}(1,\ z), \tag{26}$$

$$r\frac{\partial \overset{k+1}{\omega}}{\partial r}(r,\ z) \to 0,\ r \to 0, \tag{27}$$

which is obtained from the nonlinear problem (20)-(23).

3) Let us set $\overset{k+1}{v}(r,\ z) = \overset{k+1}{\omega}(r,\ z) + \overset{k}{V}(r,\ z)$ and let us find the (k + 1)-th approximation for S by means of this approximation, etc.

The iteration process is performed until the necessary accuracy ε is reached in determining the function v(r, z), i.e., until satisfaction of the condition

$$|\overset{k+1}{v}(r,\ z) - \overset{k}{v}(r,\ z)| < \varepsilon |\overset{k}{v}(r,\ z)|,\ (r,\ z) \in D.$$

Problem (24)-(27) was solved numerically by using finite-difference methods. An internal iteration process was used with inversion of the Laplace difference operator by using a direct method [9] on the basis of a fast Fourier transform. Computation of the approximate location of the phase interface S was performed by using linear interpolation in the values of v at the mesh nodes. The procedure of calculating the temperature u by means of the values of the auxiliary function v in the case when k(u) is a piecewise-constant function with a discontinuity at u = u* raises no difficulties. In the more general case the computation of u is performed at each k-th iteration by using interpolation in a sufficiently detailed table of values of $v(u_i)$, i = 1, 2, ..., M under the assumption that the function k(u) is linear in each segment $[u_i,\ u_{i+1}]$.

RESULTS OF THE COMPUTATIONS

The following heat conduction and specific heat coefficients are used in the examples presented below:

$$k(u) = \begin{cases} 0.6u + 0.4, & u < 1, \\ 1, & u \geqslant 1, \end{cases} \quad c(u) = \begin{cases} 0.15u + 0.63, & u < 1, \\ 1, & u \geqslant 1. \end{cases}$$

Here u_0 = 0.15 in the boundary conditions (8), (9); heat source intensity is selected in the form

$$Q(z) = 1\Big/\sqrt{1 + \left(\frac{z - z_0}{r_0}\right)^8},$$

where z_0 = 0.5L and r_0 = 0.25.

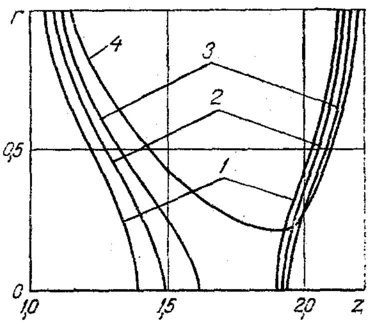

Fig. 3.

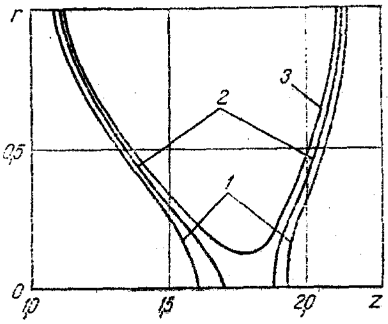

Fig. 4.

Fig. 3. Dependence of the melt boundary on the Stefan number (Pe = 0.5; Kr = 1.0): 1) St = 0; 2) 0.125; 3) 0.25; 4) 0.5.

Fig. 4. Melt boundary with heat transfer to the environment taken into account (St = 0.25; Pe = 0.5, Kr = 1.0): 1) Bi = 0; 2) 0.25; 3) 0.5.

The computations were performed on a (33 × 33) mesh in the rectangle D with L = 3. The selection of such an L is due to the necessity that the influence of the side boundaries does not affect the melting isotherm. The numerical experiments performed displayed rapid convergence of the proposed iteration process and good accuracy of the developed method in the test problems.

The solution of problems with a varying Kirpichev number is presented in Fig. 1. Let us note that the thermal source intensity at Kr = 0.5 does not suffice to melt the rod. The influence of the Peclet number is shown in Fig. 2. As should have been expected, as Pe increases, corresponding to an increase in the velocity of rod motion, the domain of the melt diminishes and shifts in the direction of rod motion. An analogous pattern of thermal front behavior is observed for a change in the Stefan number (Fig. 3).

The melting isotherms are displayed in Fig. 4 for problems with the parameters Pe = 0.5, St = 0.25, and Kr = 1, in which, in addition, the heat transfer to the external medium is taken into account. This corresponds to the fact that the condition $k(u)(\partial u/\partial r)(1, z) = KrQ(z) - Bi\,\alpha(u)(u - u_0)$ is satisfied instead of (9) in the problem (6)-(10). In our example

$$\alpha(u) = \begin{cases} u - 0{,}125, & u < 1, \\ 1, & u \geqslant 1, \end{cases}$$

and the Biot number changes. Taking account of the heat losses associated with radiation is performed analogously.

NOTATION

(r, φ, z), cylindrical coordinates; R, radius of the cylindrical rod; v_0, velocity of rod motion; u^*, melting point; u_0, temperature of the environment; Q(z), heat flux on the rod side boundary; λ, enthalpy of the phase transition; k(u), heat conduction; c(u), specific heat; $\alpha(u)$, coefficient of heat transfer to the environment; $Pe = v_0Rc_0/k_0$, Peclet number; $St = \lambda/c_0u^*$, Stefan number; $Kr = Q_0R/k_0u^*$, Kirpichev number, and $Bi = \alpha_0R/k_0$, Biot number.

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